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Vibratory instability of cellular flames propagating in tubes

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In this paper, we study the vibratory instability of a cellular flame, propagating downwards in a tube, which results from the coupling between the longitudinal acoustic modes of the tube and the modification of the cellular flame structure by the acceleration of the acoustic field. We assume that the wrinkling of the flame is of small amplitude a_0, which is the case when the flame burning velocity is just above the critical velocity characterizing the Darrieus–Landau instability threshold. We demonstrate that, in this case, the growth rate of the corresponding thermoacoustic instability, non-dimensionalized with the acoustic frequency, is proportional to $(k_c a_0)^2$, where k_c is the critical wavenumber of the cellular instability. If one extends the result up to amplitudes of the same order as the wavelength, then one obtains a relative growth rate of order unity which is much larger than the one obtained from the study of the vibratory instability of the planar flame. As is observed in experiments, the theory predicts that the primary sound is generated when the amplitude of the cells is sufficiently large that the fundamental tone becomes unstable first and that the vibratory instability for the fundamental tone occurs in the lower half of the tube. This suggests that the coupling between cellular flame and acoustic field studied here is the mechanism for primary sound generation.

1. Introduction

The understanding of vibratory instabilities of flames propagating in tubes is of great importance in explaining the spontaneous generation of primary sound which can occur during flame propagation. In a typical experiment (Searby 1991), the mixture is ignited at the open top of a vertical tube. In a first stage the flame propagates downwards with a cellular shape. Then, when the flame reaches the centre of the tube, primary sound is generated. The acoustic energy increases and the cellular structure disappears. The flat flame oscillates in the acoustic field generated by the first instability. Then a parametric instability occurs, i.e. the flame wrinkles again with a cell amplitude oscillating at half of the acoustic excitation frequency. Finally the amplitude of the cells increases until a turbulent combustion regime occurs.

The parametric instability is now well understood (Markstein 1970) and has been clearly confirmed experimentally (Searby & Rochwerger 1991). The important problem which remains unsolved is the generation of primary sound and more precisely the particular mechanism of the corresponding instability. The general theory for thermoacoustic instabilities was developed by Rayleigh in 1878 and can be summarized by the following criterion: when heat is released locally and periodically in a gaseous medium, an acoustic oscillation is amplified if the oscillating component of pressure and heat flux are in phase. For reviews on this subject, see Putnam (1964) and Strehlow (1979). However, for a particular problem the exact nature of the coupling between heat sources and acoustic field needs to be identified in order to characterize completely the instability. Three basic mechanisms can lead to a constructive interaction between acoustic field and flame structure: (i) the direct effect of the pressure and temperature of incoming acoustic waves on the flame burning velocity (Dunlap 1950); (ii) the penetration of the flame edge into the acoustic boundary layer (Kaskan 1953); (iii) the variations of flame area caused by the acceleration of the acoustic velocity field (Markstein 1970 and Rauschenbakh 1961).

The first mechanism has been analysed in detail by Clavin, Pelcé & He (1990). They determine that the growth rate of the corresponding thermoacoustic instability, non-dimensionalized with the acoustic frequency, is proportional to βM where β is the dimensionless activation energy of the limiting chemical reaction and M the Mach number. It is shown that in the range of parameters where the flat flame is stable ($M \approx 10^{-3}$), the direct coupling between pressure variations in the acoustic wave and flame structure is weak ($\beta M \approx 10^{-2}$) and may bearly overcome the damping mechanism due to the radiation of acoustic waves from the open ends of the tube and by thermal and viscous dissipation at the tube walls. From an experimental point of view, the conclusions are less clear. The reason is that the range of parameters for which the flat flame is stable with respect to shape deformations is very narrow. Flames propagating with a burning velocity larger than 15 cm/s become cellular. Flames with a velocity lower than 8 cm/s extinguish. In this range of small burning velocities, in agreement with theory, no acoustic instability has been observed (Searby 1991). Thus comparison between theory and experiment for this mechanism is inconclusive: in the range where theory predicts thermoacoustic instability of a flat flame, the flame is already cellular.

Little is known from a theoretical point of view about the second mechanism except that heat flux variations are localized in the acoustic boundary layer. In the bulk, the flat flame oscillates with the growing acoustic field and, close to the wall, due to the penetration of the flame edge into the boundary layer, the flame shape remains almost at rest in the frame moving with the mean flame velocity. As a result, the part of the flame which penetrates into the boundary layer elongates periodically at the acoustic frequency and may act as a heat source interacting constructively with the acoustic field. It is expected that when the boundary-layer thickness is smaller than the flame dead space, which is the case at sufficiently large frequency, this instability mechanism dies out.

As mentioned by Markstein (1970), the third mechanism would be the most important because in the conditions where primary sound is generated, flames are sufficiently fast to be cellular (Searby 1991). A preliminary analysis of this mechanism was performed by Rauschenback (1961). By using a simplified model for flame dynamics he determined that the growth rate of the corresponding thermoacoustic instability non-dimensionalized with the acoustic frequency was proportional to $(a_0 k)^2$, where k is the wavenumber of the cellular flame and a_0 the amplitude of the cells. Thus, for cellular flames of amplitude of the same order as the wavelength the dimensionless growth rate of the instability is of order unity and thus larger by a factor 100 than the one found for the first mechanism. The main purpose of the paper is to improve the analysis of this mechanism in order to make possible comparisons between theory and experiment. For this, we use a more complete model for flame dynamics which was found to be successful for the determination of the stability limits of the Darrieus–Landau instability (Pelcé & Clavin 1982; Quinard, Searby & Boyer 1985) and introduce gravity effects into the problem in order to analyse a situation close to the cellular instability threshold, where the flame cells have a small amplitude, and thus in a range of control parameters where calculations can be done analytically.

First we present the development of the complete analysis and evaluate the orders of magnitude of the different physical quantities. Then we develop the flame model and explain the basic assumptions that are used. Thirdly, the calculation of the transfer function, which is the most important technical part of the analysis, is performed analytically in the limit where the steady cellular flame has a small amplitude compared to the wavelength of the cells. Then, we determine the growth rate of the corresponding thermoacoustic instability and discuss the result found in the light of experimental observations.

2. Development of the analysis

All the analysis is based on the fact that the heat source, the flame, is localized at some position in the tube and interacts with acoustic longitudinal modes, as it is observed in experiments. In this case, the flame appears effectively as a piston interacting with the acoustic modes of the tube. When the interaction is constructive, instability occurs and acoustic energy is self-generated in the tube. The main problem is to determine the characteristics of this effective piston, that is pressure and longitudinal velocity jumps across the discontinuity surface, in order to determine the stability limits of the corresponding vibratory instability.

Consider first the pressure jump across the flame: we assume that the flame propagates at small Mach number, which is the case for most experimental observations reported. Then, the pressure jump across the flame is of order ρU^2, where U is the flame velocity and is much smaller than the pressure variations in the acoustic wave ρc^2, where c is the velocity of sound. Thus, to a first approximation, pressures can be assumed equal on both sides of the flame. This is equivalent to the effective piston having a negligible mass, so that during its motion, forces exerted by fresh and burned mixtures on the piston are equal.

Consider now the longitudinal velocity jump across the flame: when a piston is non-porous so that the flow cannot penetrate the discontinuity surface, longitudinal flow velocities are equal on both sides of the discontinuity surface and equal to the piston velocity. The case of the flame is in general different. Flow can penetrate the discontinuity surface so that a jump of longitudinal velocity is generated. The corresponding relative jump, called in the following the transfer function $\mathscr{T} = (\delta u_2(0) - \delta u_1(0))/\delta u_1(0)$, where the subscripts 1 and 2 identify respectively the fresh and burned mixtures, is determined by an analysis of the flame response to an external acoustic field δu_1. To evaluate the order of magnitude of this transfer function consider a cellular flame of small amplitude a_0 and wavenumber $k, z = \bar{\zeta}(x) = a_0 \cos kx$. Assume that the cellular flame is perturbed by an incoming acoustic wave in the fresh gases of amplitude $\delta u_1(0)$ and frequency ω. Introduce the corresponding perturbation of the flame amplitude δa_0. Because of mass conservation across the flame the amplitude of the outgoing wave in the burned gases is determined by the relation

$$\delta u_2(0) - \delta u_1(0) = \left(\frac{\rho_1}{\rho_2} - 1\right) u_{\mathrm{L}}\, \delta S, \tag{1}$$

where u_L is the flame burning velocity and $\delta S = 2\pi k^2 a_0 \, \delta a_0$ the perturbation of the flame area. To evaluate δa_0, notice that the acceleration of the acoustic field causes a pressure jump across the cellular flame of magnitude $\delta\pi \approx (\rho_1 - \rho_2)(\mathrm{d}(\delta u_1)/\mathrm{d}t))\, a_0$ is generated which induces a local flow around the cells of magnitude $\delta u \approx \delta\pi/\rho_1 u_L$. If one assumes a constant relative flame burning velocity, this secondary flow generates a flame shape deformation of amplitude $\delta a_0 = \delta u / \mathrm{i}\omega$ and thus, from (1), a relative jump of acoustic velocities across the discontinuity proportional to $(a_0 k)^2$.

It appears that this factor is related to the relative area of the unperturbed cellular flame: $S = 1 + \frac{1}{4}(a_0 k)^2$. The basic reason for this may be understood from the Rayleigh criterion which states that thermoacoustic instability results from a favourable coupling between heat release and acoustic pressure fluctuations in the tube. The rate of heat released per unit cross-sectional area of the tube by the cellular flame is $q = \rho_1 u_L c_p (T_2 - T_1) S$, where T_1 and T_2 are the temperatures of fresh and burned mixtures respectively. It follows that the fluctuations of heat release are proportional to variations of flame area δS, which are proportional to $(a_0 k)^2$ when the amplitude of the cellular flame is weak. Thus one can expect that the relative jump of acoustic velocities across the discontinuity, which will determine the strength of the thermoacoustic instability, is also proportional to this factor.

Then the growth rate of the instability can be determined after solving the classical acoustic problem of the longitudinal modes with the above-mentioned boundary conditions on the flame and additional conditions at the tube extremities. Acoustic losses are first neglected in the analysis. The ratio between amplitudes of the velocity of outgoing and incoming waves is determined as a function of the relative position of the flame in the tube. Then the use of the transfer function relation allows the amplitude of the acoustic wave to be eliminated and the so-called eigenmode equation to be obtained whose solutions are the possible acoustic frequencies of the tube. If one takes $\mathscr{T} = 0$ in the eigenmode equation (no velocity jump across the flame) one obtains the classical free eigenmodes of a tube with a free piston separating two media of different densities. When the transfer function is small, as it is the case for wrinkled flames of small amplitudes, one can expand the eigenmodes around the free eigenmodes and obtain the growth rate of the thermoacoustical instability as the real part of the perturbed eigenvalue. At the first order in the expansion, the growth rate non-dimensionalized with the acoustic frequency is found to be proportional to the transfer function and thus to $(a_0 k)^2$.

The instability can develop only when the linear growth rate evaluated above is sufficiently large to overcome damping effects. In principle, these effects can be introduced in the analysis (see for instance Clavin *et al.* 1990). There are two reasons why we here evaluate only their order of magnitude. First, in experiments, the factor $(a_0 k)^2$ is of order unity and thus much larger than the magnitude of the acoustic losses. Second, wavelengths of cellular flames produced in experiments are dependent on the initial conditions so that exact stability limits are difficult to obtain experimentally.

There are two kinds of damping effects: heat transfer and viscous friction at the tube wall, and acoustic radiation losses at the open end of the tube. In order to obtain the order of magnitude of the first mechanism, one evaluates the damping rate of the acoustic energy due to viscous friction (in gases the Prandtl number is of order unity, so the damping rate due to heat conduction to the wall is of the same magnitude as viscous damping). If δu is the amplitude of the acoustic velocity in the tube, the energy dissipated in the boundary layer per unit time and unit length of the tube is of order of $\rho\nu(\delta u)^2 R/h$, where $h \approx (\nu/\omega)^{\frac{1}{2}}$ and R is the tube radius. The acoustic energy stored in the acoustic mode per unit length of the tube is $\rho(\delta u)^2 R^2$. It follows that the corresponding damping rate is $1/\tau_{\mathrm{dv}} \approx (\nu\omega)^{\frac{1}{2}}/R$. Another damping mechanism is due to the release of acoustic energy from the open end of the tube. The order of magnitude of the corresponding damping rate can be determined in the following way. The acoustic power radiated by an oscillating body of volume $V(t)$ is

$$\mathrm{d}E_{\mathrm{r}}/\mathrm{d}t \approx \rho(\mathrm{d}^2V/\mathrm{d}t^2)^2/c.$$

Here the oscillating body is the column of gas close to the tube exit which enters and comes out of the tube periodically. The corresponding volume of gas is $V(t) = R^2\omega\delta u$ so that the rate of acoustic energy lost by radiation satisfies $\mathrm{d}E_{\mathrm{r}}/\mathrm{d}t = -1/\tau_{\mathrm{dr}} E_{\mathrm{r}}$ where $1/\tau_{\mathrm{dr}} \approx (R/\lambda)^2$ is the corresponding damping rate and λ the acoustic wavelength. These rates increase with the harmonic number, so that the least damped mode is the fundamental tone, which is expected to be the first mode to become unstable, as has been observed in experiments. If the damping due to the radiative losses dominates, which is the case in all but very long thin tubes, then the vibratory instability occurs for the fundamental tone $\omega \approx c_1/L$, where L is the length of the tube and c_1 the sound velocity in the fresh mixture, if the linear growth rate mentioned above dominates $1/\tau_{\mathrm{dr}}$, i.e. in order of magnitude:

$$a_0 k \geqslant R/L. \tag{2}$$

3. The model

When a cellular flame propagates at small Mach number in a tube (figure 1) whose diameter is much smaller than its length, two different regions can be distinguished: (i) an acoustic region, outside the flame, where the flow is dominated by the one-dimensional acoustic waves of the order of $\lambda = c/\omega$, and (ii) a region, called the 'inner region' in the following, located around the flame, of thickness Λ, the wavelength of the cellular flame, where the flow can be assumed to be incompressible. The ratio of the sizes of these two regions, $\Lambda/\lambda \approx M\omega\Lambda/u_{\mathrm{L}}$, is effectively a small parameter when the Mach number of the flame is small.

Consider first the incompressible region in a frame of reference where the flame is at rest, i.e. moving with respect to the laboratory frame with the velocity $U = \mathrm{d}z_0/\mathrm{d}t$. Here $z_0(t)$, which defines the new origin of the frame, is the average in space of the location of the flame. In this frame and in a region of size of order Λ around the flame, the velocity field w of the flow satisfies the incompressible Euler equation

$$\nabla\cdot \boldsymbol{w} = 0 \tag{3a}$$

for the conservation of mass, and

$$\rho(\partial \boldsymbol{w}/\partial t + (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{w}) = -\nabla\pi \tag{3b}$$

for the dynamics. Here $\pi = p - \rho(g + \mathrm{d}U/\mathrm{d}t)\,z$ is the effective pressure, p the pressure, ρ is equal to ρ_1 in the fresh gas and to ρ_2 in burned gas, and g is the magnitude of the acceleration due to gravity assumed positive when the flame propagates downwards. At the interface, the following four boundary conditions must be satisfied:

$$\rho_1(\boldsymbol{w}_{-\mathrm{f}} - \boldsymbol{v}_{\mathrm{i}})\cdot\boldsymbol{n} = \rho_2(\boldsymbol{w}_{+\mathrm{f}} - \boldsymbol{v}_{\mathrm{i}})\cdot\boldsymbol{n} \tag{4a}$$

for mass conservation;

$$\boldsymbol{w}_{-\mathrm{f}}\cdot\boldsymbol{\tau} = \boldsymbol{w}_{+\mathrm{f}}\cdot\boldsymbol{\tau} \tag{4b}$$

for the equality of tangential velocities;

$$\pi_{-\mathrm{f}} + \rho_1(g + \mathrm{d}U/\mathrm{d}t)\,z + \rho_1((\boldsymbol{w}_{-\mathrm{f}} - \boldsymbol{v}_{\mathrm{i}})\cdot\boldsymbol{n})^2 = \pi_{+\mathrm{f}} + \rho_2(g + \mathrm{d}U/\mathrm{d}t)\,z + \rho_2((\boldsymbol{w}_{+\mathrm{f}} - \boldsymbol{v}_{\mathrm{i}})\cdot\boldsymbol{n})^2 \tag{4c}$$

for conservation of the momentum component normal to the interface; and

$$(\boldsymbol{w}_{-\mathrm{f}} - \boldsymbol{v}_{\mathrm{i}})\cdot\boldsymbol{n} = u_{\mathrm{L}}. \tag{4d}$$

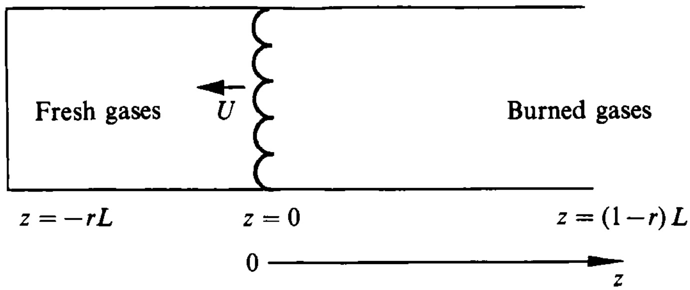

FIGURE 1. General configuration.

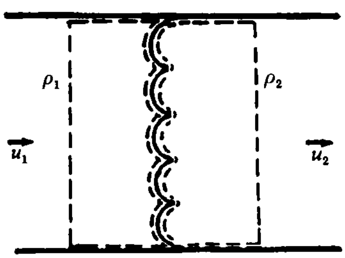

FIGURE 2. Contours of integration in the incompressible zone.

Here $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are respectively the normal and tangent directions to the interface. For simplicity we first neglect the curvature effects of the flame, i.e. we assume that the flame is an infinitely thin discontinuity surface propagating with constant velocity. This simplifying assumption allows us to discuss more clearly the reduction of the inner problem to a linear response problem. Then, curvature terms are reintroduced for the complete calculation of the transfer function.

Now consider the acoustic region where, as a first step, the mean position of the flame is kept at rest in the laboratory frame by adjusting the mean flow velocity of the incoming fresh gases to the mean velocity of the flame. We first neglect all effects leading to dissipation of acoustic energy in the tube, i.e. viscous friction, conduction of heat to the tube walls and acoustic losses at the open end of the tube. Then the longitudinal acoustic field can be written simply as

$$\delta p_{1,2} = \{A_{1,2} \exp(\mathrm{i}\,\omega/c_{1,2} z) + B_{1,2} \exp(-\mathrm{i}\,\omega/c_{1,2} z)\} \exp(\mathrm{i}\omega t), \tag{5a}$$

$$\delta u_{1,2} = -1/\rho_{1,2} c_{1,2} \{A_{1,2} \exp(\mathrm{i}\,\omega/c_{1,2} z) - B_{1,2} \exp(-\mathrm{i}\,\omega/c_{1,2} z)\} \exp(\mathrm{i}\omega t), \tag{5b}$$

where subscripts 1 and 2 denote unburned and burned mixtures respectively. At this spatial scale, the inner incompressible region appears as a discontinuity surface separating fresh and burned mixtures. As was mentioned in the introduction, the fluctuations of acoustic pressures can be considered as equal on both sides of the discontinuity, since the flame propagates at small Mach number:

$$\delta p_1(0) = \delta p_2(0). \tag{6}$$

To determine the jump conditions of velocity field across the discontinuity surface, one applies the mass conservation across the inner region, i.e. integrates the incompressibility condition over two volumes delimited by the contours shown on figure 2. After integration of the incompressibility condition over the volume corresponding to the fresh mixture, one obtains the relative flame velocity as

$$u_1 - U = u_L S, \tag{7}$$

where S is the surface of the flame relative to the area of the cross-section of the tube. After integration of the same equation for the second volume and with the help of the local mass conservation relation (3a), one obtains the global mass conservation relation as

$$\rho_1(u_1 - U) = \rho_2(u_2 - U). \tag{8}$$

The combination of these relations allows us to obtain the jump of the acoustic field across the flame as

$$\delta u_2(0) - \delta u_1(0) = (\rho_1/\rho_2 - 1)\, u_L\, \delta S. \tag{9}$$

The feedback between a cellular flame and acoustics is now clear. From (7) the flame velocity is determined from the acoustic velocities. Then, (4c) determines how the flame shape and thus the surface is affected by the acceleration. Last, (8) determines how the acoustic velocities are modified by the variation of the flame surface.

We consider the classical configuration where the flame propagates from the open to the closed end of a tube so that the appropriate boundary conditions at the tube extremities are closed end in the fresh mixture

$$z = -rL: \quad \delta u_1 = 0; \tag{10a}$$

open end in the burnt gases

$$z = (1-r)L: \quad \delta p_2 = 0. \tag{10b}$$

4. Order of magnitude and approximations

The complete problem is difficult to solve because when posed in the inner region it is in general nonlinear and time dependent. However, the problem is simplified if one considers weakly cellular flames. Assume first that the flame is flat. Then, (4c) indicates that the flame shape is not affected by the acceleration ($z = 0$). The flame surface is constant and no acoustic energy is generated by the flame during its oscillation in the given acoustic field. Consequently the jump of acoustic velocities across the discontinuity vanishes and the flame behaves as an ordinary piston, i.e. it oscillates with the velocity $\delta U = \delta u_1 = \delta u_2$. Assume now that the flame is weakly cellular, $z = \bar{\xi}(x)$. Then, (4c) shows that a hydrodynamic pressure jump across the flame $\delta\tilde{\pi} = (\rho_1 - \rho_2)(\mathrm{d}(\delta u_1)/\mathrm{d}t)\,\bar{\xi}(x)$ is generated which induces a local flow around the cells. This secondary flow modulates the flame shape and thus generates a jump of acoustic velocities across the discontinuity from relation (8). Thus, it is convenient to look for a solution in the inner region such as

$$\xi(x) = \bar{\xi}(x) + \tilde{\xi}(x), \tag{11a}$$

$$\pi_{1,2}(x,z) = p_1 - \rho_{1,2}(g + \mathrm{d}(\delta u_1)/\mathrm{d}t)\,z + \rho_{1,2}\, u_{\mathrm{L,B}}^2 + \bar{\pi}_{1,2} + \tilde{\pi}_{1,2}, \tag{11b}$$

$$w_{1,2z}(x,z) = u_{\mathrm{L,B}} + \bar{w}_{1,2z} + \tilde{w}_{1,2z}, \tag{11c}$$

$$w_{1,2x}(x,z) = \bar{w}_{1,2x} + \tilde{w}_{1,2x}, \tag{11d}$$

where $u_\mathrm{B} = (\rho_1/\rho_2)\, u_\mathrm{L}$. The $\bar{w}$ are associated with the weakly cellular steady solution. Their order of magnitude relative to the steady planar flow is related to ϵ, the ratio of the amplitude and wavelength of the cellular flame, assumed here to be a small parameter. The $\tilde{w}$ are associated with the secondary flow generated around the cells by the acoustic perturbation. As shown by (4c), their order of magnitude is related to the amplitude of δu_1 the planar acoustic disturbance, relative to the velocity u_L of the steady planar flow. In (3) terms involving the non-uniform steady flow are small compared to the dominant term and can be neglected in a first approximation. For instance, $\bar{w}_{1z}\,\partial\tilde{w}_{1z}/\partial z$ and $\tilde{w}_{1z}\,\partial\bar{w}_{1z}/\partial z$ are negligible compared to $u_\mathrm{L}\,\partial\tilde{w}_{1z}/\partial z$.

Similarly, nonlinear terms in the boundary conditions involving the steady cellular shape are negligible compared to the linear one. For instance, in the boundary condition for the tangential velocities, the term $\bar{w}_{1z}\,\partial\tilde{\xi}/\partial z$ is much smaller than $u_L\,\partial\tilde{\xi}/\partial x$. Thus, with the assumption that the steady flame is weakly cellular, the inner problem appears as a linear response problem, i.e. the perturbation of the flame shape is a solution of a linear equation with a non-homogeneous term determined by the acoustic forcing. It follows that at leading order in ϵ, the flow generated by the acoustic perturbation satisfies the equations

$$\rho_{1,2}(\partial\tilde{w}_{1,2z}/\partial t + u_{L,B}\,\partial\tilde{w}_{1,2z}/\partial z) = -\partial\tilde{\pi}_{1,2}/\partial z, \tag{12a}$$

$$\rho_{1,2}(\partial\tilde{w}_{1,2x}/\partial t + u_{L,B}\,\partial\tilde{w}_{1,2x}/\partial z) = -\partial\tilde{\pi}_{1,2}/\partial x, \tag{12b}$$

$$\partial\tilde{w}_{1,2x}/\partial x + \partial\tilde{w}_{1,2z}/\partial z = 0, \tag{12c}$$

with the following boundary conditions on the non-perturbed planar interface ($z = 0$):

$$\tilde{w}_{1z} = \tilde{w}_{2z}, \tag{13a}$$

$$\tilde{w}_{1x} + u_L\,\partial\tilde{\xi}/\partial x = \tilde{w}_{2x} + u_B\,\partial\tilde{\xi}/\partial x, \tag{13b}$$

$$\partial\tilde{\xi}/\partial t - \tilde{w}_{1z} = 0, \tag{13c}$$

and

$$\tilde{\pi}_1 + \rho_1 g\tilde{\xi} + \rho_1\,\mathrm{d}(\delta u_1)/\mathrm{d}t\,\bar{\xi} = \tilde{\pi}_2 + \rho_2 g\tilde{\xi} + \rho_2\,\mathrm{d}(\delta u_1)/\mathrm{d}t\,\bar{\xi}. \tag{13d}$$

In order that the time-derivative term will be conserved in (11) the forcing frequencies ω must not be too low. More precisely, this term must be larger than the nonlinear term $\bar{w}_{1z}\,\partial\tilde{w}_{1z}/\partial z$ previously neglected, or $\omega \gg \bar{w}/\Lambda$, where Λ is the wavelength of the cellular flame. Once this linear problem is solved the jump of the acoustic velocities across the discontinuity is determined by (8), i.e.

$$\delta u_2(0) - \delta u_1(0) = \left(\frac{\rho_1}{\rho_2} - 1\right) u_L \frac{1}{\Lambda}\int_0^{\Lambda} \frac{\mathrm{d}\bar{\xi}}{\mathrm{d}x}\frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}x}\,\mathrm{d}x. \tag{14}$$

In order to solve the problem further it is necessary to overcome two difficulties. First, we need the explicit steady solution for the cellular flame. With the assumption that the normal burning flame velocity is constant along the whole surface, only approximate solutions have been determined (Zeldovich *et al.* 1980). Secondly, the linear problem posed by (12) and boundary conditions (13) admits general solutions (i.e. solutions of the homogeneous problem) whose amplitude grows exponentially with time with the Darrieus–Landau growth rate. These solutions overtake the particular oscillating solution after a finite time and the linear response problem posed above loses its meaning.

A more realistic situation can be considered if one takes into account the effects of flame thickness, i.e. essentially the dependence of the burning flame velocity on flame curvature and flow stretch. It is well known that in this case a flat flame can be stable for sufficiently low velocity (Quinard *et al.* 1985). At the threshold of instability, i.e. when the flame propagates with a critical velocity u_{Lc} determined by the diffusive characteristics of the reactive mixture, the flame becomes cellular with the marginal wavenumber k_c. The shape of the flame is simply

$$\bar{\xi}(x) = a_0 \cos k_c x, \tag{15}$$

where the flame amplitude a_0 is arbitrary but small.

When effects due to flame curvature and flow stretch are taken into account, boundary conditions (13) are modified. If the typical size Λ and timescale of the wrinkling of the flame are respectively much larger than the thickness of the flame d and the transit time d/u_L, one expands the boundary conditions (13) in $\epsilon = d/\Lambda$ and obtains, up to order ϵ^2 (Pelcé & Clavin 1982):

$$\tilde{w}_{2z}-\tilde{w}_{1z} = du_L\left(\frac{\rho_1-\rho_2}{\rho_2}Ma-\frac{\rho_1}{\rho_2}\log\left(\frac{\rho_1}{\rho_2}\right)\right)\left(\frac{1}{u_L}\frac{\partial\tilde{w}_{1z}}{\partial z}-\frac{\partial^2\tilde{\xi}}{\partial z^2}\right), \tag{16a}$$

$$\tilde{w}_{2x}-\tilde{w}_{1x} = -u_L\frac{\rho_1-\rho_2}{\rho_2}\frac{\partial\tilde{\xi}}{\partial x}+d\log\left(\frac{\rho_1}{\rho_2}\right)\left(\frac{1}{u_L}\frac{\partial\tilde{w}_{1x}}{\partial t}+\frac{\partial^2\tilde{\xi}}{\partial t\,\partial x}+\frac{\partial\tilde{\xi}}{\partial x}\frac{g}{u_L}\right)$$
$$-k_c\frac{d}{u_L}\log\left(\frac{\rho_1}{\rho_2}\right)\frac{\mathrm{d}(\delta u_1)}{\mathrm{d}t}a_0\sin(k_c x), \tag{16b}$$

$$\tilde{w}_{1z}-\frac{\partial\tilde{\xi}}{\partial t} = u_L Ma\,d\left(\frac{1}{u_L}\frac{\partial\tilde{w}_{1z}}{\partial z}-\frac{\partial^2\tilde{\xi}}{\partial z^2}\right), \tag{16c}$$

and

$$\tilde{\pi}_2+\rho_2 g\tilde{\xi}-\tilde{\pi}_1-\rho_1 g\tilde{\xi}+\rho_1 u_L^2 d\left(2\frac{\rho_1-\rho_2}{\rho_2}\left(Ma-\frac{\rho_1}{\rho_1-\rho_2}\log\left(\frac{\rho_1}{\rho_2}\right)\right)\left(\frac{1}{u_L}\frac{\partial\tilde{w}_{1z}}{\partial z}-\frac{\partial^2\tilde{\xi}}{\partial z^2}\right)\right.$$
$$\left.+\frac{\rho_1-\rho_2}{\rho_2}\frac{\partial^2\tilde{\xi}}{\partial z^2}+\log\left(\frac{\rho_1}{\rho_2}\right)\frac{d}{u_L^2}\frac{\partial\tilde{w}_{1z}}{\partial t}\right) = (\rho_1-\rho_2)\frac{\mathrm{d}(\delta u_1)}{\mathrm{d}t}a_0\cos(k_c x), \tag{16d}$$

where Ma is the dimensionless Markstein length.

5. The transfer function

We first determine the perturbation of the flame amplitude,

$$\xi = Q\cos(k_c x)\exp(\mathrm{i}\omega t)$$

in response to the fluctuating acoustic field $\delta u_1\exp(\mathrm{i}\omega t)$. For this, we eliminate the velocity field from (12) by differentiating with respect to the variable x (resp. z) of (12a) (resp. (12b)) and summing the two resulting equations. Then, the effective pressure satisfies the Laplace equation whose solution is

$$\tilde{\pi}_{1,2} = P_{1,2}\cos(k_c x)\exp(\pm k_c z)\exp(\mathrm{i}\omega t). \tag{17}$$

After integration of (11) on both sides of the discontinuity, we obtain respectively the longitudinal and transverse velocity fields:

$$w_{1z} = -\frac{P_1}{\rho_1(\mathrm{i}\omega+u_L k)}\cos(k_c x)\exp(k_c z)\exp(\mathrm{i}\omega t), \tag{18a}$$

$$\tilde{w}_{1x} = \frac{P_1}{\rho_1(\mathrm{i}\omega+u_L k)}\sin(k_c x)\exp(k_c z)\exp(\mathrm{i}\omega t), \tag{18b}$$

$$\tilde{w}_{2z} = \frac{P_2}{\rho_2(\mathrm{i}\omega-u_B k)}\cos(k_c x)\exp(-k_c z)+R\exp\left(-\frac{\mathrm{i}\omega}{u_B}z\right)\cos(k_c x)\exp(\mathrm{i}\omega t), \tag{18c}$$

$$\tilde{w}_{2x} = \frac{P_2}{\rho_2(\mathrm{i}\omega-u_B k)}\sin(k_c x)\exp(-k_c z)+\frac{\mathrm{i}\omega}{u_L k_c}R\exp\left(-\frac{\mathrm{i}\omega}{u_B}z\right)\sin(k_c x)\exp(\mathrm{i}\omega t), \tag{18d}$$

where P_1, P_2 and R are for the moment unknown complex coefficients. We assume that the energy of the acoustic field is growing in the tube due to an eventually

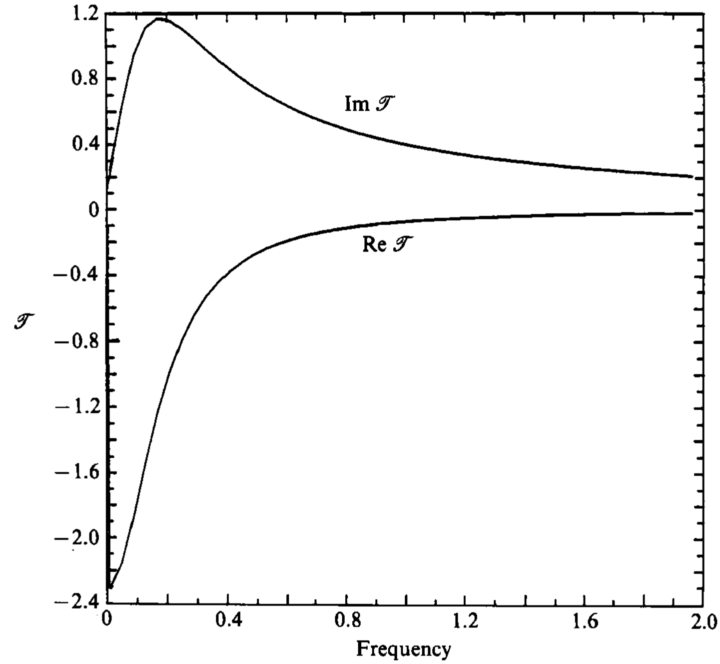

FIGURE 3. Transfer function as a function of the reduced frequency $\Omega = \omega d/u_1$ for $Ma = 3$. The corresponding critical values for the laminar flame speed and the reduced wavenumber are $u_L = 9.9$ cm/s, and $K_c = 0.0723$.

favourable coupling between the cellular flame and the acoustic field. It follows that the imaginary part of ω is negative so that we keep the term proportional to R corresponding to the general solution of (12) only on the burned side. Then we apply the boundary conditions (16) and obtain a set of four linear equations for the coefficients P_1, P_2, R and Q. After resolution of this system of equations we determine Q as

$$Q = \frac{-\mathrm{i}\Omega C(K_c)}{-\Omega^2 A(K_c) + \mathrm{i}\Omega B(K_c) + D(K_c)} a_0 \frac{\delta u_1}{u_L}, \tag{19}$$

where $\Omega = \omega d/u_L$ and $K = kd$ are respectively the dimensionless frequency and wavenumber. Here,

$$A(K) = (2-\gamma) + \gamma(Ma - 1/\gamma \log(1/1-\gamma))K, \tag{20a}$$

$$B(K) = 2K + 2/(1-\gamma)(Ma - \log(1/1-\gamma))K^2, \tag{20b}$$

$$C(K) = \gamma K(1 - K(Ma - 1/\gamma \log(1/1-\gamma)), \tag{20c}$$

and

$$D(K) = \gamma/(1-\gamma)K(gd/u_L^2(1-\gamma) - K(1 + gd/u_L^2(1-\gamma)(Ma - 1/\gamma \log(1/1-\gamma)) + K^2(1 + (2+\gamma)/\gamma Ma - 2/\gamma \log(1/1-\gamma)). \tag{20d}$$

When the denominator of the right-hand side of (12) vanishes, Ω and K are related by the dispersion relation for the disturbances of the planar flame (see (36) and (37) in Clavin *et al.* 1990). The instability threshold of the planar flame is determined by the relations $\Omega(K_c) = 0$, $\mathrm{d}\Omega/\mathrm{d}K(K_c) = 0$. It follows that $D(K_c) = 0$,

$$K_c = 2(1-\gamma)gd/u_L^2, \tag{21}$$

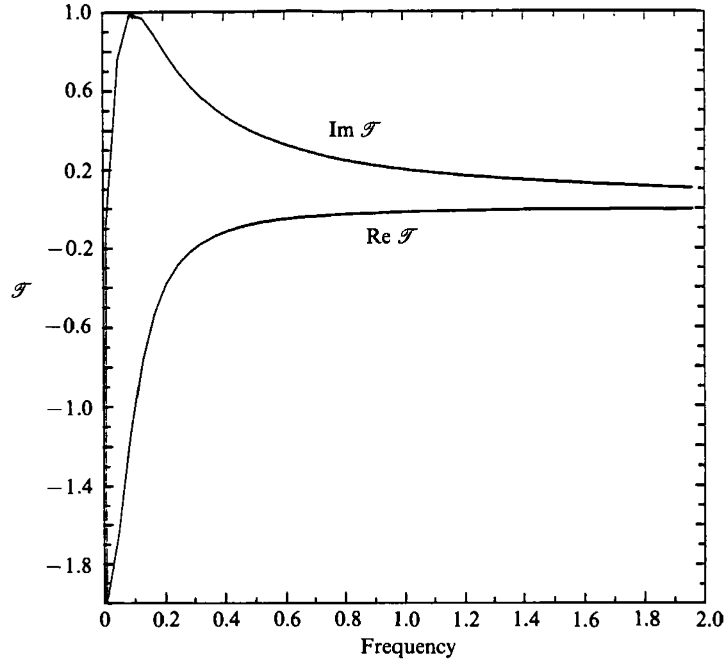

FIGURE 4. As figure 3 but for $Ma = 5$ and $u_L = 12$ cm/s, and $K_c = 0.0378$.

and
$$1/2K_c = 1+(2+\gamma)/\gamma\,(Ma-2/\gamma\log(1/1-\gamma)), \tag{22}$$

which relates the value of the marginal wavenumber to the value of the Markstein length at the instability threshold.

Then, the transfer function $\mathscr{T} = (\delta u_2(0)-\delta u_1(0))/\delta u_1(0)$ can be easily derived from (14) and (19) as
$$\mathscr{T} = \frac{1}{2}\frac{\gamma}{1-\gamma}(K_c A_0)^2\frac{-\mathrm{i}\Omega C(K_c)}{-A(K_c)\,\Omega^2+B(K_c)\,\mathrm{i}\Omega}. \tag{23}$$

It is proportional to the small parameter $(K_c A_0)^2$, where $A_0 = a_0/d$ is the dimensionless amplitude of the steady cellular flame. Real and imaginary parts of $\mathscr{T}/(K_c A_0)^2$ are drawn on figures 3 and 4 for rich and poor ethylene–oxygen mixtures which correspond to Markstein numbers $Ma = 3$ and 5 respectively (Quinard & Searby 1990). The imaginary part of the transfer function, which plays a decisive role in the following discussion on the instability criterion, is found to be positive in the whole range of forcing frequencies.

6. Vibratory instability of the cellular flame

6.1. *Instability criterion*

Applying the boundary conditions (10) at the tube ends and (5) to relate the acoustic velocity and pressure field at the flame location gives
$$u_1(0) = -\mathrm{i}/\rho_1 c_1 p_1(0)\tan(rX), \tag{24a}$$
and
$$u_2(0) = -\mathrm{i}/\rho_2 c_2 p_2(0)\cot((1-r)\,c_1/c_2 X), \tag{24b}$$

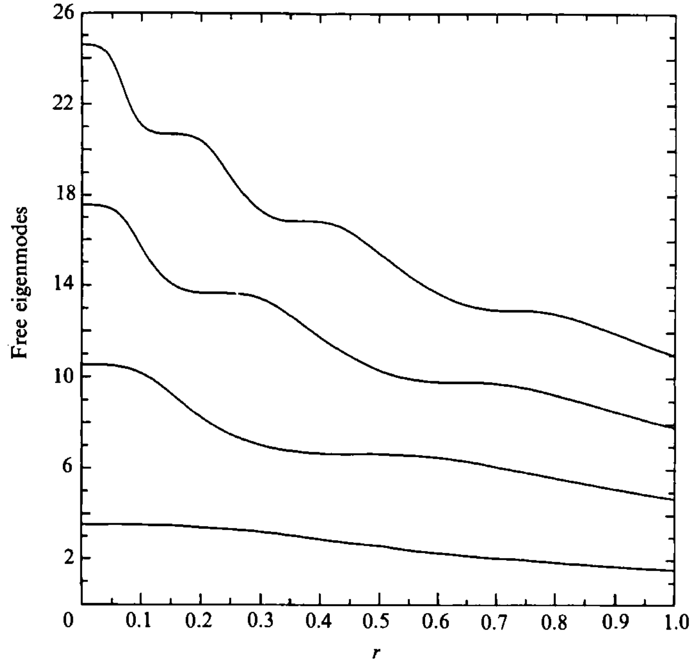

FIGURE 5. Free eigenmodes of the tube as a function of the relative position r of the flame.

where X is the dimensionless frequency $X = \omega L/c_1$. By using the boundary conditions (5) and (23) at the flame location, one finally obtains the eigenmode equation:

$$\rho_2 c_2/\rho_1 c_1 \tan(rX)\tan((1-r)\,c_1/c_2 X)(1+\mathscr{T}) = 1. \tag{25}$$

From (23), the transfer function is proportional to $(K_c A_0)^2$, which in our analysis is assumed to be a small parameter. It follows that the solutions of (25) can be expanded around the solutions for the free eigenmodes of the tube, X_0, which are given by

$$\rho_2 c_2/\rho_1 c_1 \tan(rX_0)\tan((1-r)\,c_1/c_2 X_0) = 1. \tag{26}$$

The X_0 are real numbers which characterize the acoustic frequencies of the tube when the flame is considered as a passive interface separating two different gaseous media. The corresponding eigenmodes are plotted in figure 5 for the first four harmonics. Writing $X = X_0 + \delta X$, one obtains at first order in the power expansion of $(K_c A_0)^2$

$$\mathrm{Im}\,(\delta X) = \frac{-\mathrm{Im}\,(\mathscr{T})\tan(rX_0)}{r(1+\tan^2(rX_0)) + \rho_2 c_2/\rho_1 c_1(1-r)\,c_1/c_2(1+\tan^2((1-r)\,c_1/c_2 X_0)\tan^2(rX_0)}. \tag{27}$$

As the denominator of the right-hand side of (27) is always positive, instability occurs if $\mathrm{Im}\,(\mathscr{T})\tan(rX_0)$ is positive. If the fundamental tone is considered, $\tan(rX_0)$ is positive for all positions of the flame in the tube. It follows that this mode is always unstable since the imaginary part of the transfer function is positive. As is shown on figure 6 the growth rate (27) calculated for $K_c A_0 = 1$ is maximum in the lower half of the tube. When the instability develops, the amplitude of the acoustic field grows, as the amplitude of the cellular flame. The analysis breaks down when the perturbed flame amplitude becomes of the same order as the flame amplitude itself. As mentioned in the first section the growth rate of the instability must overcome the

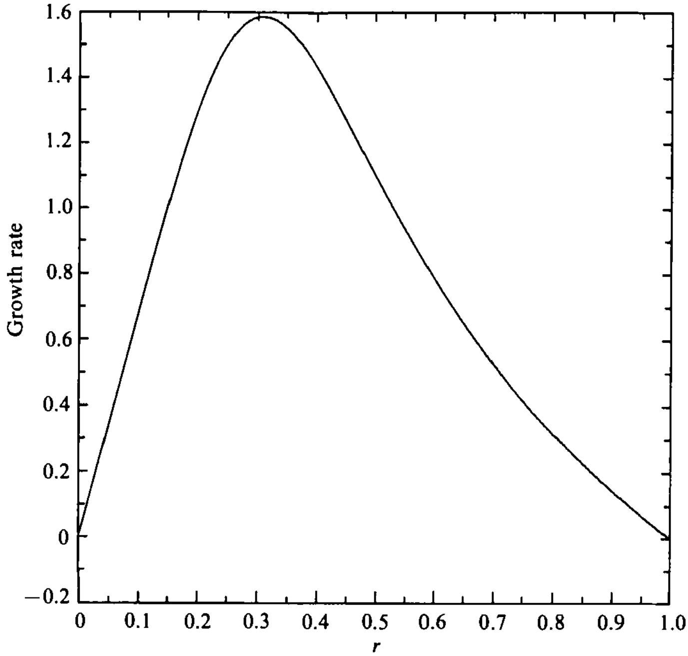

FIGURE 6. Growth rate of the vibratory instability as a function of the relative position r of the flame in the tube for $Ma = 3$.

damping effects due to the heat transfer and viscous friction at the tube wall. The exact damping rate can be computed in principle in a similar way to that in Clavin *et al.* (1990). However, note when comparing theory with experiments, that the growth rate (27) contains the undetermined factor $(a_0 k)^2$ which is difficult to control experimentally, and a precise calculation of the damping effects will be too refined for the uncertainties of the experiment. One needs to remember simply instability criterion (2), i.e. that the cellular thermoacoustic instability occurs if the amplitude of the wrinkled flame is sufficiently large.

6.2. *Comparison with experiments*

Before discussing the instability criterion (2) let us recall the experimental observations (Searby 1991): (i) for sufficiently weak mixtures, corresponding to laminar flame speeds below about 15 cm/s, the flame propagates to the bottom of the tube producing no sound; (ii) for laminar speeds greater than 15–20 cm/s 'low-intensity' acoustic oscillations occur when the flame is in the lower half of the tube; (iii) for laminar flame speeds above 25 cm/s the flame accelerates suddenly, producing 'high'-intensity acoustic oscillations.

The theory developed here concerns essentially (i) and (ii), i.e. the generation of primary sound in the tube. The reactive mixture considered here is lean propane–air mixture for which the Markstein number $Ma \approx 4.5$. The corresponding critical velocity for the Darrieus–Landau instability is around 11 cm/s. It follows that below 15 cm/s the flame is still weakly cellular, and thus weakly unstable for the vibratory instability since the corresponding growth rate is proportional to $(K_c A_0)^2$. In this case damping effects dominate and no sound is produced. For a larger velocity, the flame shape modulation is of larger amplitude and the destabilizing effect can overcome damping effects. Considering the growth rate of the destabilizing effect

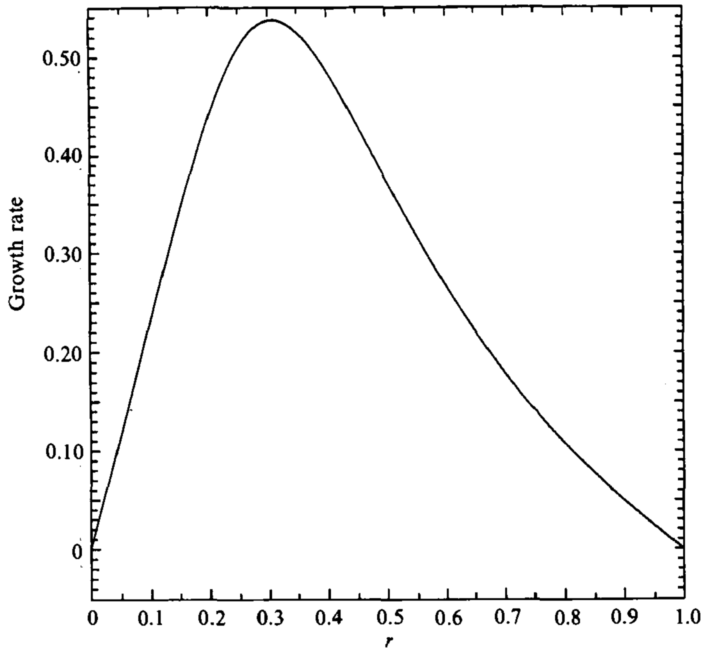

FIGURE 7. As figure 6 but for $Ma = 5$.

versus the relative position r in the tube drawn on figure 7, it appears that the maximum growth rate occurs in the lower half of the tube, which is in agreement with observation (ii). Thus, at least qualitatively, theory is in agreement with experimental observations. Quantitative disagreement on the stability limits (27) still remains. It is clear that the main discrepancies between theory and experiment originate from the flame model. Flames are assumed here to be of small amplitude; in experiments their amplitude is of order unity. Acoustics in the cavity can in principle be computed very precisely (see for instance Clavin *et al.* 1990). In order to achieve a quantitative agreement, two further pieces of work are needed. From a theoretical point of view, the transfer function for cellular flames with a relative amplitude of order unity needs to be computed. This appears for the moment a difficult problem if solved with analytical tools. From an experimental point of view, cellular flames of well defined wavelength need to be produced.

7. Conclusion

We have determined the acoustic transfer function for a weakly cellular flame, i.e. when the propagation conditions are close to the threshold of the Darrieus–Landau instability. It is found that this transfer function is proportional to $(k_c a_0)^2$, where k_c and a_0 are respectively the wavenumber and the amplitude of the wrinkling of the flame at the instability threshold. This is multiplied by a frequency factor whose denominator is the dispersion relation for disturbances of a planar flame front, as is usual for linear response problems. The consequences are that, as is observed in experiments: (i) the primary sound is generated when the amplitude of the spontaneous cellular structure of the flame is sufficiently large; (ii) the fundamental tone is the most unstable mode; (iii) vibratory instability for the fundamental tone occurs in the lower half of the tube. Some quantitative disagreement still remains on the stability limits of the vibratory instability.

Thus the mechanism of variation of the flame area caused by the acceleration of the acoustic velocity field appears to be a good candidate for the explanation of the generation of primary sound by flames propagating in tubes.

Much work is still needed to validate this scenario quantitatively: one has to control experimentally the wavelength of the cellular flame; and one needs to determine transfer functions for cellular flames with a relative amplitude of wrinkling of order unity.

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